Tutorial Note XII

1 Relations between Three Kinds of Convergence

There are three kinds of convergence mentioned in our course: uniform convergence, pointwise convergence, and L^2 convergence. Roughly speaking, they could be listed from strong to weak: uniform convergence, pointwise convergence, and L^2 convergence. The meanings of the first two kinds of convergence are as their names, and the meaning of the L^2 convergence could be understood as average convergence. Next we investigate their relations. First, it is easy to see that uniform convergence implies pointwise convergence and L^2 convergence. Conversely, it does not hold. An example is as follows:

$$f_n(x) = 1_{(0,1/n]}.$$

It is easy to see that $f_n \to 0$ pointwisely and in L^2. However, f_n do not converge uniformly to 0. For the relation between pointwise convergence and L^2 convergence, real analysis provides us the following propositions:

- (DCT) If $|f_n| \leq M$ on $[0,1]$ and $f_n \to f$ pointwisely, then $f_n \to f$ in L^2;
- (Riesz) If $f_n \to f$ in L^2, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ almost everywhere.

But in general, they can't imply each other. For pointwise convergence $\not\Rightarrow$ L^2 convergence, an example is as follows:

$$f_n = n1_{(0,1/n^2]}.$$

It is easy to see that $f_n \to 0$ pointwisely, however, $\|f_n\|_2 = 1$ and thus $f_n \nrightarrow 0$ in L^2. For L^2 convergence $\not\Rightarrow$ pointwise convergence, an example is as follows:

$$f_1 = 1_{(0,1]};$$

$$f_2 = 1_{(0,1/2]}, \quad f_3 = 1_{(1/2,1]};$$

$$f_4 = 1_{(0,1/3]}, \quad f_5 = 1_{(1/3,2/3]}, \quad f_6 = 1_{(2/3,1]}$$

$$\cdots.$$

It is easy to see that $f_n \to 0$ in L^2, however f_n do not converge at any point in $(0, 1]$.

2 Energy Estimate by Fourier Series

In this section, we derive the energy estimates by Fourier series. Consider the following IBVP:

$$\begin{cases} u_{tt} - u_{xx} = 0; \\ u(0,t) = u(1,t) = 0. \end{cases}$$

By the method of separation of variables, we could write u as

$$\sum_n (A_n \cos n\pi t + B_n \sin n\pi t) \sin n\pi x.$$

If we allow to differentiate term by term, we have

$$\begin{aligned} u_t &= \sum_n n\pi(-A_n \sin n\pi t + B_n \cos n\pi t) \sin n\pi x, \\ u_x &= \sum_n n\pi(A_n \cos n\pi t + B_n \sin n\pi t) \cos n\pi x. \end{aligned}$$

Then by Parseval's identity,

$$\begin{aligned} \int_0^1 (u_t^2 + u_x^2) &= \frac{1}{2} \sum_n (n\pi)^2 [(-A_n \sin n\pi t + B_n \cos n\pi t)^2 + (A_n \cos n\pi t + B_n \sin n\pi t)^2] \\ &= \frac{1}{2} \sum_n (n\pi)^2 (A_n^2 + B_n^2). \end{aligned}$$

So the energy estimate is proved by Fourier series.

3 Decay of Fourier Coefficients

In this section, we discuss decay of Fourier series. In principle, for Fourier transforms, we have the following correspondence:

$$\text{regularities} \longleftrightarrow \text{decay}.$$

For Fourier series, some results are as follows:

- If f is α-Hölder continuous, $\hat{f}(n) = O(1/|n|^\alpha)$;
- If f is bounded monotone, $\hat{f}(n) = O(1/|n|)$;
- If f is continuous, $\hat{f}(n) = o(1)$;
- If f is C^k, $\hat{f}(n) = o(1/|n|^k)$.

Here we just present the proof of the third one, which is the Riemann-Lebesgue lemma. The method we employ is the stationary phase method and density argument. In fact, we will prove it for $f \in L^1$. We begin with $f \in C^\infty[0, 2\pi]$. For

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nx} f(x) \,\mathrm{d}x, \tag{1}$$

the idea of the stationary phase method is to use

$$(-\mathrm{i}n)^{-k} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathrm{e}^{-\mathrm{i}nx} = \mathrm{e}^{-\mathrm{i}nx}$$

to perform integration by parts. The key is that after performing integration by parts we will obtain decay. In fact, the stationary phase method is used to deal with asymptotic problems of oscillatory integrals like

$$\int \mathrm{e}^{\mathrm{i}\lambda\varphi(x)} \psi(x) \,\mathrm{d}x,$$

where φ is called the phase. If there is no stationary point of the phase, by integration by parts, we will obtain decay. If there is a stationary point, we will need more effort to deal with it. An intuition behind the stationary phase method is as follows: for (1), there are a lot of cancellations hidden in $\mathrm{e}^{-\mathrm{i}nx}$, that is,

$$\int_{x_0}^{x_0 + 2\pi/n} \mathrm{e}^{-\mathrm{i}nx} \,\mathrm{d}x = 0,$$

and integration by parts could make use of these cancellations. Here we only need to perform integration by parts once, then we get

$$\hat{f}(n) = \int_0^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}nx}}{2\pi \mathrm{i}n} f'(x) \,\mathrm{d}x.$$

So

$$|\hat{f}(n)| \le \frac{1}{2\pi n} \int_0^{2\pi} |f'(x)| \,\mathrm{d}x$$

and $\hat{f}(n) = o(1)$.

Next we generalize the decay to L^1 by density argument. Here we use a fact that $C^\infty[0, 2\pi]$ is dense in L^1, that is, for every function $f \in L^1$ and $\varepsilon > 0$, there is a $g \in C^\infty[0, 2\pi]$ such that

$$\int_0^{2\pi} |f - g| < \varepsilon.$$

Moreover, we need to note that

$$|\hat{f}(n)| \le \frac{1}{2\pi} \int_0^{2\pi} |f|.$$

For $f \in L^1$, fix ε and take such a g. Then

$$\begin{aligned} |\hat{f}(n)| &\le |(f - g)\widehat{\ }(n)| + |\hat{g}(n)| \\ &\le \frac{1}{2\pi} \left(\int_0^{2\pi} |f - g| \right) + |\hat{g}(n)| \end{aligned}$$

$$\leq \frac{\varepsilon}{2\pi} + |\hat{g}(n)|.$$

It follows that

$$\overline{\lim_{n\to\infty}} |\hat{f}(n)| \leq \frac{\varepsilon}{2\pi}.$$

Let $\varepsilon \to 0$, then we have

$$\lim_{n\to\infty} |\hat{f}(n)| = 0.$$